JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 29, No. 2, March–April 2006

Application of the Cayley Form to General Spacecraft Motion

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The study of N-dimensional rigid-body motion is a well-developed field of mechanics. Some of the key results for describing the kinematics of these bodies come from the Cayley transform and the Cayley-transform kinematic relationship. Additionally, several forms of the equations of motion for these bodies have been developed by various derivations. By using Cayley kinematics, the motion of general mechanical systems can be intimately related to the motion of higher-dimensional rigid bodies. This is done by associating each point in the configuration space with an N-dimensional orientation. An example of this is the representation of general orbital and attitude motion of a spacecraft as pure rotation of a four-dimensional rigid body. Another example is the representation of a multibody satellite system as a four-dimensional rigid body.

Introduction

WHEREAS THE STUDY of mechanics has been motivated by the desire to explain the three-dimensional, physical universe, the mathematical models that have resulted are in no way limited to three dimensions. Higher-dimensional bodies can be kinematically defined, and by assuming that principles such as conservation of angular momentum and Hamilton's principle apply in higher-dimensional spaces, their dynamics can also be developed. In particular, one can consider the mechanics of an N-dimensional rigid body, which can be defined as a system the configuration of which can be completely defined by an $N \times N$ proper orthogonal matrix.

The descriptions of N-dimensional bodies are not simply mathematical curiosities: they can be used to describe real systems. This is done by linking the motion of general systems to the rotation of a higher-dimensional rigid body. Three-dimensional analogs to this approach have been used in the past. For example, Junkins and Turner developed an analogy between spacecraft orbital motion and rigid-body rotations.[1] In that work, a physical reference frame was defined using the spacecraft position and velocity vectors. The orbital motion could then be studied by describing the evolution of this frame. Because of the osculation constraint implied in the definition of this frame, however, its motion does not fully capture the orbital dynamics. The approach required explicit reintroduction of Newton's second law to describe the behavior of the radial distance. Additionally, whereas the kinematic analogy to a rigid body is clear, dynamically it was found that the gyroscopic equations contained variable inertia attributable to changes in the radial distance.

This paper presents a new analogy, called the Cayley form,[2] between the combined attitude and orbital motion of a spacecraft and rotational motion of a four-dimensional, rigid body. In addition to incorporating both the attitude and orbital motion, the new analogy more fully incorporates the dynamics in a general sense (i.e., osculation constraints are not imposed nor are explicit reintroductions required). Incorporating both attitude and orbital motion in a single dynamic representation could be seen as a disadvantage, because it does not take advantage of the decoupling that occurs between these two motions for the special case of unforced dynamics. Many other choices for representing translational and rotational motion, however, also exhibit coupling in the motion variables, for example, the body components of translational velocity.[3] The disadvantage is mitigated, however, by the fact that for most spacecraft systems the attitude and orbital motions are in fact coupled by both naturally occurring forcing terms, such as the rigid-body gravity potential, and control terms, such as fixed-direction thrusters. A second example is presented to further illustrate the new analogy. It involves the attitude motion of a satellite containing three momentum wheels that is also related to the rotational motion of a four-dimensional body.

The main motivation for the Cayley form, which is still under investigation, is to provide a new set of motion variables to be used in the design of controllers for nonlinear systems. Establishing global results for nonlinear control is in general very difficult. Behavior must typically be established on a system-by-system basis or for each value of initial conditions and system parameters. Perhaps one of the most-studied systems in nonlinear control, however, is spacecraft attitude motion. By relating the motion of general nonlinear systems to rotational motion, the Cayley form provides the possibility of leveraging this work in spacecraft attitude control to broader classes of problems. This is perhaps easiest to conceive of in applying conventional attitude controllers to other three degree-of-freedom (DOF) systems. An example of this in Ref. 4 demonstrated stabilization of a three-link manipulator using angular velocity feedback of the associated rotational motion defined by the Cayley form. It was shown that this controller, designed using the Cayley variables, had superior performance to other controllers designed using traditional motion variables. Of course, the Cayley form relates more complicated systems to higher-dimensional rotations. Applying the proposed approach for controller design to these systems requires generalization of attitude controllers to higher dimensions.[4] The advantage of this approach is to provide a common geometric framework for the design of controllers for generic nonlinear systems. This paper presents the necessary first step of relating spacecraft motion to higher-dimensional rotations.

In each of the examples covered in this paper, the relations are made by associating each point in the six-dimensional configuration space with a particular orientation in four-dimensional space. Similar to the Junkins and Turner analogy, the new analogy does not match all of the dynamics properties associated with rigid bodies. In the following sections of this paper the concepts of N-dimensional kinematics and dynamics are reviewed and their relationship to general systems is discussed. This is then used to analyze general spacecraft motion. First, however, some mathematical preliminaries are covered.

The Numerical Relative Tensor χ^j_{ik}

Index notation is a useful shorthand notation for manipulating matrix operations. Matrices or higher-order tensors are expressed

Presented as Paper 2004-182 at the AAS/AIAA Spaceflight Mechanics Meeting, Maui, HI, 8–12 February 2004; received 7 April 2004; revision received 17 April 2005; accepted for publication 25 April 2005.

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Table 1 Summary of Properties Related to χ_{ik}^{j}

Property	Value
Generating vector to skew-symmetric matrix	$U_{ik} = \chi_{ik}^{j} u_j$
Skew-symmetric matrix to generating vector	$u_j = \frac{1}{2}\chi_{ik}^{j} U_{ik}$
Skew-symmetry in the lower indices	$\chi_{jk}^{j} = -\chi_{ki}^{j}$
Upper-index identity	$\chi_{ik}^{j}\chi_{ik}^{l} = 2\delta_{jl}$
χ-δ identity	$\chi_{jk}^{i}\chi_{mn}^{i} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$
Lower-index identity	$\chi_{jk}^{i}\chi_{jn}^{i} = (N-1)\delta_{kn}$

as $\boldsymbol{A}$, and their elements are expressed using subscript indices. For example, the elements of a matrix $\boldsymbol{A}$ are A_{ij}. The Einstein summation convention is that if any index is repeated twice within a term, then the term represents the summation for every possible value of the index. An index must not be repeated more than twice in a term. Indices that appear only once in each term of an equation are free indices, and the equation is valid for each possible value of the index. The Kronecker delta, δ_{ij}, is equal to unity if $i = j$ and is equal to zero otherwise.

A new three-index, numerical relative tensor, χ_{ik}^{j}, was introduced in an earlier work[5] to relate the elements of an $N \times N$ skew-symmetric matrix to an M-dimensional vector form, where $M = N(N-1)/2$. For example, consider the angular-velocity matrix $\boldsymbol{\Omega}$ and vector $\boldsymbol{\omega}$:

$$\Omega_{ik} = \chi_{ik}^{j}\omega_j \tag{1}$$

The upper index j in this expression is summed from one to M, whereas the lower indices i and k take on values from one to N. In the familiar $N = 3$ case, χ_{ik}^{j} becomes the Levi–Civita permutation symbol ϵ_{ijk}.[5,6] The vector $\boldsymbol{\omega}$ in Eq. (1) is the generating vector of the skew-symmetric matrix $\boldsymbol{\Omega}$. The elements of $\boldsymbol{\omega}$ are related to the elements of $\boldsymbol{\Omega}$ in the following form:

$$[\boldsymbol{\Omega}] = \begin{bmatrix} 0 & -\omega_M & \cdots & & & & \cdots \\ \omega_M & 0 & & & & & \cdots \\ \vdots & & \ddots & & & & \cdots \\ & & & 0 & -\omega_6 & \omega_5 & -\omega_4 \\ & & & \omega_6 & 0 & -\omega_3 & \omega_2 \\ \vdots & \vdots & \vdots & -\omega_5 & \omega_3 & 0 & -\omega_1 \\ & & & \omega_4 & -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{2}$$

This N-dimensional form matches the three-dimensional form and leads to the following expression for χ_{ik}^{j}:

$$\chi_{ik}^{j} = \begin{cases} (-1)^{i+k}\operatorname{sgn}(k-i) & \text{for} \quad j = z \quad \text{and} \quad i \neq k \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where

$$x = \tfrac{1}{2}\left(i + k + \sqrt{(k-i)^2}\right), \qquad y = \tfrac{1}{2}\left(i + k - \sqrt{(k-i)^2}\right)$$

$$z = \tfrac{1}{2}y^2 + \left(\tfrac{1}{2} - N\right)y - x + \tfrac{1}{2}N^2 + \tfrac{1}{2}N + 1$$

Several properties of χ_{ik}^{j} are summarized in Table 1.[5]

Cayley Kinematics

Rotations in N dimensions are described by a proper, orthogonal matrix $\boldsymbol{C} \in \mathrm{SO}(N)$ called a rotation matrix. The Cayley transform is a remarkable set of relationships between these proper orthogonal matrices and a skew-symmetric representation.[7] Cayley discovered the forward relationship while investigating some properties of "left systems."[8]

Forward:

$$\boldsymbol{C} = (\boldsymbol{I} - \boldsymbol{Q})(\boldsymbol{I} + \boldsymbol{Q})^{-1} = (\boldsymbol{I} + \boldsymbol{Q})^{-1}(\boldsymbol{I} - \boldsymbol{Q}) \tag{4}$$

Inverse:

$$\boldsymbol{Q} = (\boldsymbol{I} - \boldsymbol{C})(\boldsymbol{I} + \boldsymbol{C})^{-1} = (\boldsymbol{I} + \boldsymbol{C})^{-1}(\boldsymbol{I} - \boldsymbol{C}) \tag{5}$$

Here, $\boldsymbol{Q}$ is an $N \times N$ skew-symmetric matrix, and $\boldsymbol{I}$ is the identity matrix. The matrix $\boldsymbol{Q}$ comprises a set of M distinct parameters that represent the orientation of an N-dimensional reference frame. In fact $\boldsymbol{q}$, the generating vector of $\boldsymbol{Q}$, is the vector of extended Rodrigues parameters (ERPs) for N-dimensional spaces whose values vary from $+\infty$ to $-\infty$.[9,10] The description of orientation using M ERPs is a demonstration that rotations in N dimensions have M degrees of freedom. Only for the special case $N = M = 3$ does the dimension of the space equal the number of rotational degrees of freedom.

The inverse form of the Cayley transform is singular for matrices $\boldsymbol{C}$ associated with one or more principal rotations equal to $\pm\pi$ rad.[11] Because $\boldsymbol{Q}$ can be related to $\boldsymbol{q}$ through the tensor $\boldsymbol{\chi}$, this inverse form can be seen as a surjective map from $\mathrm{SO}(N)$ onto $\mathrm{R}(M)$. The forward form is a map from the global space $\mathrm{R}(M)$ onto the subset of $\mathrm{SO}(N)$ not related to principal rotations equal to $\pm\pi$ rad.

For N dimensions the angular velocity is defined through the evolution of the rotation matrix:

$$\boldsymbol{\Omega} = \boldsymbol{C}^{-1}\dot{\boldsymbol{C}} \tag{6}$$

In general, the angular velocity is a set of quasi velocities related to rotational motion.[12] Combining the angular velocity with the Cayley transform produces the Cayley-transform kinematic relationships. These relationships connect the derivatives of the M independent parameters of $\boldsymbol{Q}$ to the angular-velocity matrix[13]:

$$\boldsymbol{\Omega} = 2(\boldsymbol{I} + \boldsymbol{Q})^{-1}\dot{\boldsymbol{Q}}(\boldsymbol{I} - \boldsymbol{Q})^{-1} \tag{7}$$

$$\dot{\boldsymbol{Q}} = \tfrac{1}{2}(\boldsymbol{I} + \boldsymbol{Q})\boldsymbol{\Omega}(\boldsymbol{I} - \boldsymbol{Q}) \tag{8}$$

Equations (7) and (8) represent a linear mapping between the generalized velocities $\dot{\boldsymbol{Q}}$ and a set of quasi velocities $\boldsymbol{\Omega}$ for N-dimensional rotations. These equations involve operations on $N \times N$ matrices, but they can be rewritten in the more familiar M-dimensional vector form:

$$\dot{q}_i = A_{im}\boldsymbol{\omega}_m \tag{9}$$

This was carried out by Sinclair and Hurtado[2] using the numerical relative symbol $\boldsymbol{\chi}$. The elements of $\boldsymbol{A}$ are given as a function of $\boldsymbol{Q}$ and the symbol $\boldsymbol{\chi}$ in the following:

$$A_{im} = \tfrac{1}{2}\left(\delta_{im} - \chi_{vp}^{i}\chi_{vl}^{m}Q_{lp} - \tfrac{1}{2}\chi_{vp}^{i}\chi_{kl}^{m}Q_{vk}Q_{lp}\right) \tag{10}$$

N-Dimensional Rigid-Body Dynamics

From the kinematics of N-dimensional bodies, the question of studying the dynamics of N-dimensional bodies naturally arises. This was first suggested by Cayley.[8] The equations of motion for rotations of N-dimensional bodies were first found by Frahm,[14] who approached the problem by considering the motion of a collection of particles in N-dimensional space. More recently geometric methods have been used to describe the free evolution of the angular-momentum matrix $\boldsymbol{L}$ of an N-dimensional body.[15–18] That work has resulted in an elegant form for the equations of motion called the Lax pair form.

The equations of motion for N-dimensional rigid bodies can also be achieved using Lagrange's equations in terms of quasi velocities. One method for doing this uses Hamel coefficients.[5] The Hamel coefficients for N-dimensional rotations are independent of the generalized coordinates and are given by the following:

$$\gamma_{vr}^{m} = \tfrac{1}{2}\chi_{ik}^{m}\left(\chi_{ic}^{v}\chi_{ck}^{r} - \chi_{ck}^{v}\chi_{ic}^{r}\right) \tag{11}$$

These values are used in Lagrange's equations to produce the vector form of the equations of motion as shown in Table 2, where T is the kinetic energy as a function of the generalized coordinates and angular velocity, and g_k is given below:

$$g_k \equiv A_{rk}\left(f_r + \frac{\partial T}{\partial q_r}\right) \tag{12}$$

Here f_r are the generalized forces and include potential and nonpotential forces. An alternative choice would be to write the equations of motion in terms of the Lagrangian and include only nonpotential

Table 2 Dynamics of *N*-dimensional rigid bodies

Form	Kinematics/Dynamics
M-dimensional vector	$\dot{q}_i = A_{ij}\omega_j$ $\mathrm{d}(\partial T/\partial\omega_k)/\mathrm{d}t + \frac{1}{2}\chi^r_{ij}(\chi^k_{ic}\Omega_{cj} - \chi^k_{cj}\Omega_{ic})(\partial T/\partial\omega_r) = g_k$
$N\times N$ matrix	$\dot{\boldsymbol{Q}} = \frac{1}{2}(\boldsymbol{I}+\boldsymbol{Q})\boldsymbol{\Omega}(\boldsymbol{I}-\boldsymbol{Q})$ $\dot{\boldsymbol{L}} = [\boldsymbol{L}, \boldsymbol{\Omega}] + \boldsymbol{G}$

terms in the generalized forces. Another method for deriving the equations of motion arises from choosing the ERPs as generalized coordinates.[2] The Cayley kinematics can then be used directly to derive the dynamic equations. Both methods produce the vector form of the equations of motion shown in Table 2. Additionally, the tensor $\boldsymbol{\chi}$ can be used to map the vector form into the classic Lax pair matrix form using $\boldsymbol{g}$ as the generating vector for the matrix $\boldsymbol{G}$.

Whereas the kinematics and dynamics equations given in Table 2 were originally derived to describe the motion of N-dimensional rigid bodies, they can also be applied to any M-DOF mechanical system.[2] This leads to the interesting result that the motion of any M-DOF mechanical system is equivalent to the pure rotation of an N-dimensional rigid body.

The M-dimensional vectors of generalized coordinates and generalized velocities, $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$, for any system can be used to generate skew-symmetric matrices $\boldsymbol{Q}$ and $\dot{\boldsymbol{Q}}$. The generalized coordinates can then be substituted into the Cayley transform to generate an orthogonal matrix $\boldsymbol{C}$ that describes the orientation of an equivalent N-dimensional rigid body. The generalized coordinates of the system are equal to the ERPs of this rigid body. Additionally, these generalized coordinates and velocities can be used to define a set of Cayley quasi velocities via the Cayley-transform kinematic relationship. This set of quasi velocities is equal to the angular velocity of the equivalent N-dimensional rigid body. The Cayley quasi velocities are therefore governed by the N-dimensional rotational equations of motion described previously. This view of Lagrangian dynamics is called the Cayley form.[2]

The simplest example of this is the mapping from one-DOF translation to one-DOF (or two-dimensional) rotation. In this case $M = 1$ and $N = 2$. This case of relating translation along a line to rotation in a plane helps illustrate the concept of the Cayley form before disscussion of more complex spacecraft motions and higher-dimensional rotations. For translation along a single coordinate axis, the single generalized coordinate is $q = x$, where x is the translational position. By applying the Cayley form, this generalized coordinate is set equal to the single "Rodrigues parameter" for rotation in a plane, $q = \tan(\theta/2)$, where θ is the rotation angle. Therefore, every possible translational position is associated with a unique orientation, and translations to $\pm\infty$ are related to the rotational-parameter singularity at $\theta = \pm\pi$. In the Cayley form, the angular velocity of the rotation is used as a quasi velocity for the translation. For $N = 2$ the Cayley kinematics simplify to $\omega = -2\dot{q}/(1+q^2) = \dot{\theta}$ or $\dot{q} = -\omega(1+q^2)/2$. The conventional equation of motion for the translation subject to an applied force f is $m\ddot{q} = f$. By directly substituting the kinematics, the equation of motion for the Cayley quasi velocity is found to be $\dot{\omega} = \omega^2 q - 2f/[m(1+q^2)]$.

General Spacecraft Motion

A large variety of mechanical systems can be studied using the Cayley form. This section presents an example of the combined orbital and rotational motion of a spacecraft acted on by a gravity-gradient torque. For this problem, a spherically symmetric Earth with gravitational parameter $\mu = 398{,}600.4415$ km^3/s^2 is assumed. Additionally, higher-order effects such as higher-order gravitational torques and the influence of attitude on the orbital motion are ignored. The satellite to be considered will have a circular orbit with a radius of 8000 km.

The general motion of a three-dimensional rigid body has six DOF and can therefore be related to the pure rotation of a four-dimensional rigid body. In fact, the general motion of any N-dimensional body can be viewed as pure rotation of an $(N+1)$-dimensional body. The study of a three-dimensional rigid body using the Cayley form is slightly complicated by the fact that three-dimensional rotational motion is conventionally studied in terms of the three-dimensional angular velocity, already a set of quasi velocities. Therefore, this portion of the problem must first be converted back to a generalized velocity expression. Throughout the example, primes are used to denote three-dimensional rotational variables, as opposed to the four-dimensional Cayley variables.

The generalized coordinates for the problem are $[\boldsymbol{q}] = [s'_1\ s'_2\ s'_3\ x_1\ x_2\ x_3]^T$, where s'_1, s'_2, and s'_3 are the three-dimensional modified Rodrigues parameters (MRPs) and x_1, x_2, and x_3 are the Cartesian position coordinates with respect to an Earth-centered inertial reference frame $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ in units of Earth radii (ER = 6378 km). The choice of MRPs is somewhat arbitrary, and any three-parameter attitude representation could be used equally well. The MRPs relate the orientation of a body-fixed frame $(\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3)$ to the local vertical local horizontal (LVLH) frame. In this example the LVLH frame is defined with $\hat{\boldsymbol{a}}_1$ aligned with the radial direction, $\hat{\boldsymbol{a}}_3$ oriented out of the orbital plane, and $\hat{\boldsymbol{a}}_2$ completing the orthogonal triad. The LVLH frame was chosen as a reference to facilitate simulation over complete orbits while avoiding MRP singularities.

In terms of the velocity $\boldsymbol{v}$ (ER/s) and the angular velocity of the body with respect to the inertial reference $\boldsymbol{\omega}'$ (rad/s), the kinetic energy is given by the following, where $\boldsymbol{J}'$ is the three-dimensional inertia tensor and m is the body's mass:

$$T = \tfrac{1}{2}\boldsymbol{\omega}'^T\boldsymbol{J}'\boldsymbol{\omega}' + \tfrac{1}{2}m\boldsymbol{v}^T\boldsymbol{v} \tag{13}$$

Both $\boldsymbol{\omega}'$ and $\boldsymbol{J}'$ are coordinatized in the body frame. For the current example the following inertia properties were used:

$$[\boldsymbol{J}'] = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 200 \end{bmatrix} \mathrm{kg\,m^2} \tag{14}$$

For this choice of the inertia tensor the gravity-gradient stable configuration occurs at alignment of the $\hat{\boldsymbol{b}}_1$ and $\hat{\boldsymbol{a}}_1$ vectors.

The angular velocity of the spacecraft with respect to the inertial reference is the sum of the angular velocity with respect to the LVLH frame $\breve{\boldsymbol{\omega}}'$ and the angular velocity of the LVLH frame with respect to the inertial frame $\bar{\boldsymbol{\omega}}'$. To simplify the following developments the known solution for the angular velocity of the LVLH frame in a circular orbit will be used directly:

$$\boldsymbol{\omega}' = \breve{\boldsymbol{\omega}}' + \bar{\boldsymbol{\omega}}' = \breve{\boldsymbol{\omega}}' + n\boldsymbol{C}_1\hat{\boldsymbol{a}}_3 \tag{15}$$

Here $\boldsymbol{C}_1$ is the rotation matrix from the LVLH to the body frame and is a function of the first three generalized coordinates, and n is the mean motion of the orbit. This expression is used to expand the kinetic energy:

$$T = \tfrac{1}{2}\breve{\boldsymbol{\omega}}'^T\boldsymbol{J}'\breve{\boldsymbol{\omega}}' + \breve{\boldsymbol{\omega}}'^T\boldsymbol{J}'\bar{\boldsymbol{\omega}}' + \tfrac{1}{2}\bar{\boldsymbol{\omega}}'^T\boldsymbol{J}'\bar{\boldsymbol{\omega}}' + \tfrac{1}{2}m\boldsymbol{v}^T\boldsymbol{v} \tag{16}$$

The three-dimensional angular velocity is related to the MRP rates by the familiar kinematic differential equation:

$$\dot{\boldsymbol{s}}' = \boldsymbol{A}'\breve{\boldsymbol{\omega}}', \qquad \breve{\boldsymbol{\omega}}' = \boldsymbol{B}'\dot{\boldsymbol{s}}' \tag{17}$$

$$[\boldsymbol{A}'] = \frac{1}{4}\begin{bmatrix} 1+s_1'^2-s_2'^2-s_3'^2 & 2(s'_1s'_2-s'_3) & 2(s'_1s'_3+s'_2) \\ 2(s'_1s'_2+s'_3) & 1-s_1'^2+s_2'^2-s_3'^2 & 2(s'_2s'_3-s'_1) \\ 2(s'_1s'_3-s'_2) & 2(s'_2s'_3+s'_1) & 1-s_1'^2-s_2'^2+s_3'^2 \end{bmatrix} \tag{18}$$

$$\boldsymbol{B}' = \boldsymbol{A}'^{-1} \tag{19}$$

By using these three-dimensional kinematics, the kinetic energy can be defined in terms of the generalized coordinates and velocities:

$$\begin{aligned} T_0(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= \frac{1}{2}\dot{\boldsymbol{s}}'^T\boldsymbol{B}'^T\boldsymbol{J}'\boldsymbol{B}'\dot{\boldsymbol{s}}' + \dot{\boldsymbol{s}}'^T\boldsymbol{B}'^T\boldsymbol{J}'\bar{\boldsymbol{\omega}}' + \frac{1}{2}\bar{\boldsymbol{\omega}}'^T\boldsymbol{J}'\bar{\boldsymbol{\omega}}' + \frac{1}{2}m\boldsymbol{v}^T\boldsymbol{v} \\ &= \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{J}\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^T\boldsymbol{K} + \frac{1}{2}\bar{\boldsymbol{\omega}}'^T\boldsymbol{J}'\bar{\boldsymbol{\omega}}' \end{aligned} \tag{20}$$

$$[\boldsymbol{J}(\boldsymbol{q})] = \begin{bmatrix} \boldsymbol{B}'^T \boldsymbol{J}' \boldsymbol{B}' & \boldsymbol{0} \\ \boldsymbol{0} & m\boldsymbol{I} \end{bmatrix} \qquad [\boldsymbol{K}(\boldsymbol{q})] = \begin{bmatrix} \boldsymbol{B}'^T \boldsymbol{J}' \bar{\omega}' \\ \boldsymbol{0} \end{bmatrix} \tag{21}$$

Here, the matrix $\boldsymbol{I}$ is the 3×3 identity matrix. The linear mapping from the Cayley quasi velocities to the generalized velocities, $\dot{\boldsymbol{q}} = \boldsymbol{A}\boldsymbol{\omega}$, is found from the Cayley kinematic definition and used to write the kinetic energy in terms of the generalized coordinates and Cayley quasi velocities:

$$\begin{aligned} T_1(\boldsymbol{q}, \boldsymbol{\omega}) &= \tfrac{1}{2}\boldsymbol{\omega}^T \boldsymbol{A}^T \boldsymbol{J} \boldsymbol{A} \boldsymbol{\omega} + \boldsymbol{\omega}^T \boldsymbol{A}^T \boldsymbol{K} + \tfrac{1}{2}\bar{\boldsymbol{\omega}}'^T \boldsymbol{J}' \bar{\boldsymbol{\omega}}' \\ &= \tfrac{1}{2}\omega_i A_{li} J_{lm} A_{mj} \omega_j + \omega_i A_{li} K_l + \tfrac{1}{2}\bar{\omega}'_i J'_{ij} \bar{\omega}'_j \end{aligned} \tag{22}$$

The necessary partial derivatives of T_1 can now be taken. Note that $\boldsymbol{J}$ and $\boldsymbol{K}$ are functions of the first three generalized coordinates and are not constant:

$$\begin{aligned} \frac{\partial T_1}{\partial q_r} &= \frac{1}{2}\omega_i \left(\frac{\partial A_{li}}{\partial q_r} J_{lm} A_{mj} + A_{li} \frac{\partial J_{lm}}{\partial q_r} A_{mj} + A_{li} J_{lm} \frac{\partial A_{mj}}{\partial q_r} \right) \omega_j \\ &\quad + \omega_i \left(\frac{\partial A_{li}}{\partial q_r} K_l + A_{li} \frac{\partial K_l}{\partial q_r} \right) + \frac{\partial \bar{\omega}'_i}{\partial q_r} J'_{ij} \bar{\omega}'_j \end{aligned} \tag{23}$$

$$\frac{\partial T_1}{\partial \omega_r} = A_{lr} J_{lm} A_{mj} \omega_j + A_{lr} K_l \tag{24}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_1}{\partial \omega_k} \right) &= \dot{A}_{lk} J_{lm} A_{mj} \omega_j + A_{lk} \dot{J}_{lm} A_{mj} \omega_j + A_{lk} J_{lm} \dot{A}_{mj} \omega_j \\ &\quad + A_{lk} J_{lm} A_{mj} \dot{\omega}_j + \dot{A}_{lr} K_l + A_{lr} \dot{K}_l \end{aligned} \tag{25}$$

The derivatives of $\boldsymbol{A}$ are computed using the chain rule:

$$\dot{A}_{lk} = \frac{\partial A_{lk}}{\partial q_s} \dot{q}_s = \frac{\partial A_{lk}}{\partial q_s} A_{st} \omega_t \tag{26}$$

The partial derivatives of $\boldsymbol{A}$ are directly evaluated from Eq. (10). The derivatives of $\boldsymbol{J}$ must now be considered:

$$\frac{\partial J_{lm}}{\partial q_r} = \begin{bmatrix} \dfrac{\partial \boldsymbol{B}'^T}{\partial q_r} \boldsymbol{J}' \boldsymbol{B}' + \boldsymbol{B}'^T \boldsymbol{J}' \dfrac{\partial \boldsymbol{B}'}{\partial q_r} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}_{lm} \tag{27}$$

$$\dot{J}_{lm} = \frac{\partial J_{lm}}{\partial q_i} \dot{q}_i \tag{28}$$

The derivatives of $\boldsymbol{B}$ are computed in the following manner:

$$\frac{\partial \boldsymbol{B}'}{\partial q_r} = -\boldsymbol{B}' \frac{\partial \boldsymbol{A}'}{\partial q_r} \boldsymbol{B}' \tag{29}$$

The derivatives of $\boldsymbol{K}$ are also computed:

$$\frac{\partial K_l}{\partial q_r} = \begin{bmatrix} \dfrac{\partial \boldsymbol{B}'^T}{\partial q_r} \boldsymbol{J}' \bar{\omega}' + \boldsymbol{B}'^T \boldsymbol{J}' \dfrac{\partial \bar{\boldsymbol{\omega}}'}{\partial q_r} \\ \boldsymbol{0} \end{bmatrix}_l \tag{30}$$

$$\dot{K}_l = \frac{\partial K_l}{\partial q_i} \dot{q}_i \tag{31}$$

Finally, the partial derivatives of $\boldsymbol{A}'$ are directly evaluated from Eq. (18), and clearly only the derivatives with respect to the first three generalized coordinates are nonzero. Similarly, the partial derivatives of $\bar{\boldsymbol{\omega}}$ are evaluated by directly taking the derivatives of the rotation matrix $\boldsymbol{C}_1$ with respect to the MRP coordinates.

The generalized forces in terms of the generalized velocities are related to the moment vector $\boldsymbol{M}$ (coordinatized in the body frame) and the force vector $\boldsymbol{F}$ (coordinatized in the inertial frame) applied to the body. These are computed using the familiar gravitational models where the vector $\boldsymbol{r}$ has components consisting of the three position coordinates and a magnitude of r, and $\boldsymbol{C}_2$ is the rotation matrix from the inertial to LVLH frame. This matrix is assembled from the inertial components of $\boldsymbol{r}$ and $\boldsymbol{v}$. Additionally, a damping moment is included with the coefficient $k = 0.1\text{kg m}^2/\text{s}$ to simulate a nutation damper onboard the spacecraft:

$$\boldsymbol{M} = 3(\mu/r^3)\boldsymbol{a} \times \boldsymbol{J}'\boldsymbol{a} - k\boldsymbol{\omega}', \qquad \boldsymbol{a} = \boldsymbol{C}_1 \boldsymbol{C}_2 \boldsymbol{r}/r \tag{32}$$

$$\boldsymbol{F} = -(\mu m/r^3)\boldsymbol{r} \tag{33}$$

The generalized forces with respect to the first and fourth generalized coordinates are shown:

$$f_1 = \boldsymbol{F}^T \frac{\partial \boldsymbol{v}}{\partial \dot{q}'_1} + \boldsymbol{M}^T \frac{\partial \boldsymbol{\omega}'}{\partial \dot{q}'_1} = \boldsymbol{M}^T \boldsymbol{B}' \hat{\boldsymbol{b}}_1 \tag{34}$$

$$f_4 = \boldsymbol{F}^T \frac{\partial \boldsymbol{v}}{\partial \dot{x}_1} + \boldsymbol{M}^T \frac{\partial \boldsymbol{\omega}'}{\partial \dot{x}_1} = \boldsymbol{F}^T \hat{\boldsymbol{e}}_1 \tag{35}$$

The other generalized forces are similar to these two:

$$[\boldsymbol{f}] = [\boldsymbol{M}^T \boldsymbol{B}' \quad \boldsymbol{F}^T]^T \tag{36}$$

Using Eqs. (23–36), the equations of motion as shown in Table 2 can be assembled and solved for the components $\dot{\omega}_j$. These equations can then be integrated to directly solve for the motion of the satellite in terms of the generalized coordinates and the Cayley quasi velocities. Because of the coupling of all six DOF, however, these equations are somewhat more difficult to integrate than the traditional implementation. Solving for $\dot{\boldsymbol{\omega}}$ requires taking the inverse of a 6×6 matrix. The difficulty can be somewhat relieved through scaling. Integrating over one orbital period using the MATLAB® algorithm ODE45 required 769 integration steps for the traditional Euler and orbit equations and 989 integration steps for the Cayley form, with identical settings. Integration of the traditional implementation can also be used to verify the results from the Cayley form. The Cayley-transform kinematic relationship can be applied to the results of the traditional equations of motion to compute the quasi velocities from the generalized coordinates and velocities. The results of these numerical solutions are shown in Figs. 1 and 2.

Figure 1 shows the solutions for the generalized coordinates, three-dimensional angular velocity, and translational velocity. The simulation results represent the motion of the satellite through one complete orbit around Earth. The orbit has an inclination of 30 deg. The spacecraft is initially rotated 10 deg relative to the LVLH frame. Because of the initial conditions and the gravity-gradient torque, the spacecraft remains within 33 deg of the LVLH reference. As can be seen by s'_3 in Fig. 1, for the first 60 min the spacecraft attitude lags behind the LVLH frame because of the damping moment. As this happens the gravity-gradient moment increases and during the second 60 min the spacecraft begins to restore alignment with the

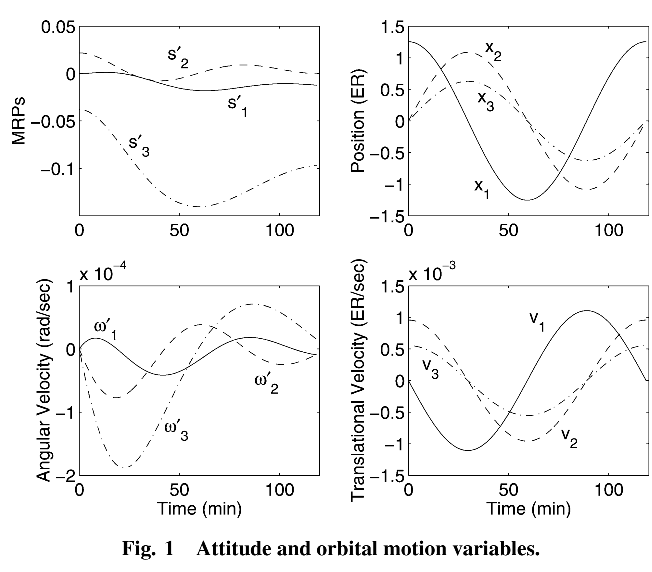

Fig. 1 Attitude and orbital motion variables.

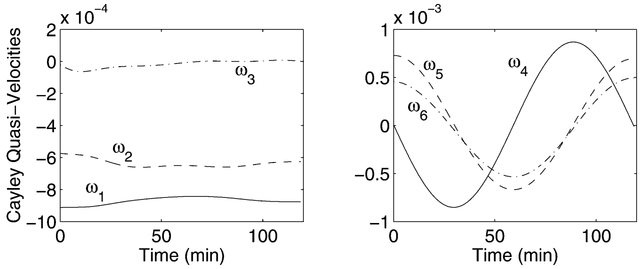

Fig. 2 Attitude and orbital motion Cayley quasi velocities.

LVLH frame. Figure 2 shows the results obtained for the Cayley quasi velocities. The implication of the Cayley form is that there is an equivalent rotational motion of a four-dimensional body that corresponds to the general motion of the satellite. For this equivalent motion the six components of the ERPs have trajectories equal to the solutions shown in Fig. 1 for the MRPs and position coordinates. The angular-velocity components are equal to the trajectories shown in Fig. 2.

These results demonstrate an interesting feature of the Cayley form having to do with the presence of singularities in the ERP description of N-dimensional orientation. Exactly analogous to the familiar Rodrigues parameter singularities, certain $N \times N$ proper orthogonal matrices result in ERP elements that tend toward infinity.[11] However, because in the Cayley form the ERPs are equal to the generalized coordinates of the original system, these rotation variables can only go to infinity if the original generalized coordinates do the same. The MRPs used in this example suffer from singularities in the conventional equations of motion; however, applying the Cayley form does not add any new singularities. For cases in which the generalized coordinates of the original problem are guaranteed to avoid divergence to infinity, the Cayley form will automatically be constrained to avoid the singular N-dimensional configurations.

Satellite with Three Momentum Wheels

The Cayley form can also be applied to multibody systems. Again, the dimension of the equivalent body in pure rotation is a function of the total number of DOF in the system. Here, the rotational motion of a satellite with three momentum wheels is considered. Similar to the previous example, this system has six DOF and is equivalent to the rotational motion of a four-dimensional body.

The generalized coordinates for this problem consist of two sets: 1) the Rodrigues parameters describing the orientation of an axes set $(\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3)$ attached to the satellite body relative to an inertial set $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ and 2) the angles describing the orientation of each wheel relative to the satellite:

$$[\boldsymbol{q}] = [q_1' \;\; q_2' \;\; q_3' \;\; \theta_1 \;\; \theta_2 \;\; \theta_3]^T \equiv [\boldsymbol{q}'^T \;\; \boldsymbol{\theta}^T]^T \tag{37}$$

The body axes attached to the satellite are assumed to be the principal axes of the satellite body. The three wheels are identical, symmetric, and aligned with the body axes. Additionally, for convenience, the wheels are assumed to be located at the system center of mass.

The kinetic energy of the system is given by the following:

$$T = \tfrac{1}{2}\boldsymbol{\omega}_b^T \boldsymbol{J}_b \boldsymbol{\omega}_b + \tfrac{1}{2}\boldsymbol{\omega}_{\mathrm{w}1}^T \boldsymbol{J}_{\mathrm{w}1} \boldsymbol{\omega}_{\mathrm{w}1} + \tfrac{1}{2}\boldsymbol{\omega}_{\mathrm{w}2}^T \boldsymbol{J}_{\mathrm{w}2} \boldsymbol{\omega}_{\mathrm{w}2} + \tfrac{1}{2}\boldsymbol{\omega}_{\mathrm{w}3}^T \boldsymbol{J}_{\mathrm{w}3} \boldsymbol{\omega}_{\mathrm{w}3} \tag{38}$$

The subscript b refers to the satellite body, and the subscripts w1, w2, and w3 refer to the three wheels. The angular velocity of wheel 1, for example, is $\boldsymbol{\omega}_{\mathrm{w}1} = \boldsymbol{\omega}_b + \dot{\theta}_1 \hat{\boldsymbol{b}}_1$. For simplicity the inertia of the satellite body was chosen as $\boldsymbol{J}_b = 100\boldsymbol{I}$ kg m^2. The axial component of inertia for each wheel was chosen to be $J_a = 10$ kg m^2, and the transverse component was $J_t = 1$ kg m^2. For convenience, the sum $\boldsymbol{J}' = \boldsymbol{J}_b + \boldsymbol{J}_{\mathrm{w}1} + \boldsymbol{J}_{\mathrm{w}2} + \boldsymbol{J}_{\mathrm{w}3}$ is defined. The kinetic energy in terms of the body angular velocity and the wheel angle rates is shown:

$$T = \tfrac{1}{2}\boldsymbol{\omega}_b^T \boldsymbol{J}' \boldsymbol{\omega}_b + J_a \boldsymbol{\omega}_b^T \dot{\boldsymbol{\theta}} + \tfrac{1}{2} J_a \dot{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}} \tag{39}$$

By using the three-dimensional kinematics, this is converted to an expression in terms of the generalized coordinates and velocities:

$$T_0 = \frac{1}{2}[\dot{\boldsymbol{q}}'^T \;\; \dot{\boldsymbol{\theta}}^T] \begin{bmatrix} \boldsymbol{B}'^T \boldsymbol{J}' \boldsymbol{B}' & J_a \boldsymbol{B}'^T \\ J_a \boldsymbol{B}' & J_a \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{q}}' \\ \dot{\boldsymbol{\theta}} \end{bmatrix} \equiv \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{J} \dot{\boldsymbol{q}} \tag{40}$$

Again, this expression is rewritten in terms of the Cayley quasi velocities using the four-dimensional Cayley kinematics:

$$T_1(\boldsymbol{q}, \boldsymbol{\omega}) = \tfrac{1}{2}\boldsymbol{\omega}^T \boldsymbol{A}^T \boldsymbol{J} \boldsymbol{A} \boldsymbol{\omega} = \tfrac{1}{2}\omega_i A_{li} J_{lm} A_{mj} \omega_j \tag{41}$$

The derivatives of T_1 are computed in a manner identical to that done in the previous example. These expressions are somewhat simpler than in the previous example because of the natural form of the kinetic energy in this problem. Another difference is that the derivatives of $\boldsymbol{J}$ now have the following form:

$$\frac{\partial J_{lm}}{\partial q_r} = \begin{bmatrix} \dfrac{\partial \boldsymbol{B}'^T}{\partial q_r} \boldsymbol{J}' \boldsymbol{B}' + \boldsymbol{B}'^T \boldsymbol{J}' \dfrac{\partial \boldsymbol{B}'}{\partial q_r} & J_a \dfrac{\partial \boldsymbol{B}'^T}{\partial q_r} \\ J_a \dfrac{\partial \boldsymbol{B}'}{\partial q_r} & \boldsymbol{0} \end{bmatrix}_{lm} \tag{42}$$

The generalized forces are computed considering an external moment $\boldsymbol{M}$ (coordinatized in the body frame) and internal moments $\boldsymbol{u}_1 = u_1 \hat{\boldsymbol{b}}_1$, $\boldsymbol{u}_2 = u_2 \hat{\boldsymbol{b}}_2$, and $\boldsymbol{u}_3 = u_3 \hat{\boldsymbol{b}}_3$. The internal moments are applied to each wheel, respectively. The external moment and internal moments $(-\boldsymbol{u} = -\boldsymbol{u}_1 - \boldsymbol{u}_2 - \boldsymbol{u}_3)$ are applied to the satellite body. The generalized forces associated with the first and fourth generalized velocities are shown:

$$\begin{aligned} f_1 &= (\boldsymbol{M} - \boldsymbol{u})^T \frac{\partial \boldsymbol{\omega}_b}{\partial \dot{q}_1'} + \boldsymbol{u}_1^T \frac{\partial \boldsymbol{\omega}_{w1}}{\partial \dot{q}_1'} + \boldsymbol{u}_2^T \frac{\partial \boldsymbol{\omega}_{w2}}{\partial \dot{q}_1'} + \boldsymbol{u}_3^T \frac{\partial \boldsymbol{\omega}_{w3}}{\partial \dot{q}_1'} \\ &= \boldsymbol{M}^T \frac{\partial \boldsymbol{\omega}_b}{\partial \dot{q}_1'} = \boldsymbol{M}^T \boldsymbol{B}' \hat{\boldsymbol{b}}_1 \end{aligned} \tag{43}$$

$$f_4 = (\boldsymbol{M} - \boldsymbol{u})^T \frac{\partial \boldsymbol{\omega}_b}{\partial \dot{\theta}_1} + \boldsymbol{u}_1^T \frac{\partial \boldsymbol{\omega}_{w1}}{\partial \dot{\theta}_1} + \boldsymbol{u}_2^T \frac{\partial \boldsymbol{\omega}_{w2}}{\partial \dot{\theta}_1} + \boldsymbol{u}_3^T \frac{\partial \boldsymbol{\omega}_{w3}}{\partial \dot{\theta}_1} = u_1 \tag{44}$$

The other generalized forces are similar to these two: $[\boldsymbol{f}] = [\boldsymbol{M}^T \boldsymbol{B}' \;\; \boldsymbol{u}^T]^T$.

By using the derivatives of T_1 and the generalized forces, the equations of motion can be assembled and solved for the components $\dot{\omega}_j$. Figures 3 and 4 show simulation results from the integration of these equations of motion and the Cayley kinematic equations. The initial conditions were chosen such that the body was initially rotating about the $\hat{\boldsymbol{b}}_1$ axis. Beginning at time zero, a constant moment of 5 Nm is applied to the second momentum wheel. The external moment was set to zero in the simulation. Figure 3 shows the solutions for the generalized coordinates, three-dimensional angular velocity, and wheel-angle rates obtained using a traditional formulation. Figure 4 shows the results obtained for the quasi velocities

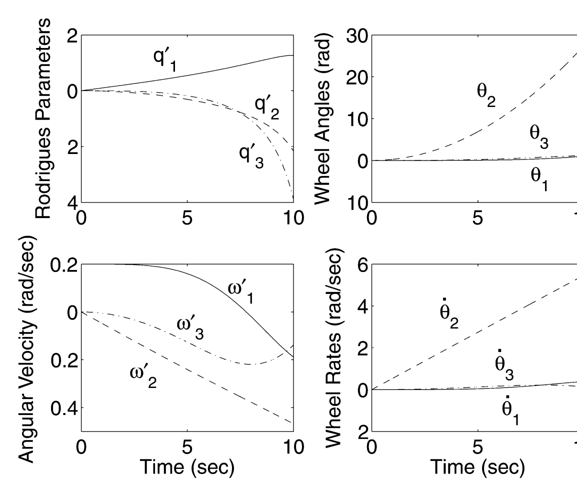

Fig. 3 Satellite-system motion variables.

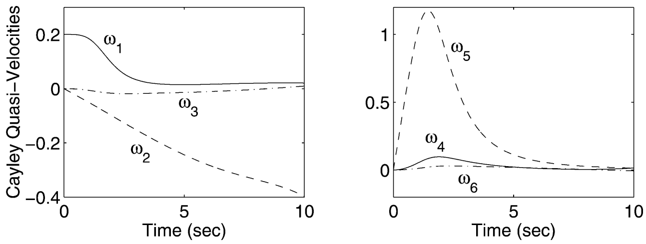

Fig. 4 Satellite-system Cayley quasi velocities.

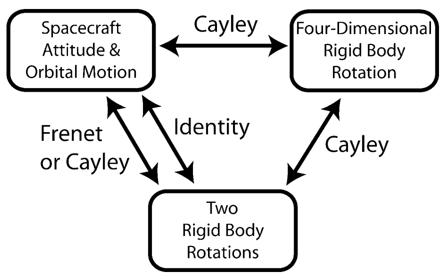

Fig. 5 Possible mappings between three equivalent systems.

using the Cayley form. Again, the implication of the Cayley form is that there is an equivalent rotational motion of a four-dimensional body that corresponds to the motion of the satellite and momentum-wheel system. For this equivalent motion the six components of the ERPs have trajectories equal to the solutions shown in Fig. 3 for the Rodrigues parameters and wheel angles. The angular velocity components are equal to the trajectories shown in Fig. 4.

Discussion

The most common representation of motion that is applied to general systems is the motion of a point in M-dimensional state space. The application of the Cayley form in this paper, however, is one example of the broad set of other possible representations. Another, more famous, example is the use of the Frenet formulas to describe particle motion in terms of a rotating reference frame, and in fact the approach of Junkins and Turner[1] is an example of this. The classic Frenet frame is defined by the velocity and acceleration vectors, whereas Junkins and Turner used position and velocity. Their analogy and the Cayley form can both be used to represent three-dimensional translations as three-dimensional rotations. Therefore, combined orbital and attitude motion, the rotation of two rigid bodies, and the rotation of a four-dimensional rigid body represent three members of a family of systems that are in a sense equivalent. Figure 5 presents a schematic of some of the possible mappings among these systems. Finally, notice that the use of a generalized coordinate vector represents each of these systems as the motion of a point in six-dimensional state space.

The Cayley form and Frenet formulas have significant differences. The Frenet formulas could potentially also be generalized to higher dimensions. Unlike the Cayley form, which associates an orientation with every point in the configuration space, the Frenet frame is defined by the generalized velocities and accelerations. In higher dimensions these vectors could form two coordinate vectors of an M-dimensional Frenet frame, leaving an $(M-2)$-dimensional orthogonal subspace for which a unique coordinatization would need to be chosen. This would then associate an M-dimensional reference frame with the M-DOF motion, again unlike the Cayley form, which uses an N-dimensional reference frame.

As mentioned, the avoidance of any osculation constraint in the Cayley form allows a more complete incorporation of the dynamics. Similar to the variable inertia term in the Junkins and Turner analogy, however, in the Cayley form the kinetic energy was found to be a function of the generalized coordinates. Although both analogies match the kinematic properties of rigid bodies, the dynamic properties of rigid bodies (i.e., constant inertia) do not precisely hold. The analogies can only go so far.

Conclusions

In this paper the kinematics and dynamics of N-dimensional bodies have been discussed and their application to general spacecraft motion has been demonstrated. This Cayley form results in new equations of motion for the orbital and attitude dynamics. A perceived disadvantage of this form is the resulting coupling that takes place between the attitude and orbital motion. This coupling, however, reveals a possible application of the Cayley form. Many special techniques have been developed for attitude determination and control in three-dimensional space. If these techniques can be generalized to higher dimensions, then the Cayley form can be used to apply them to a broader class of mechanical systems. Although not treated in this paper, some preliminary results involving feedback control using the Cayley form have been developed.

Possible applications include combined attitude and orbit determination and control. This could be of particular interest for systems that already exhibit a high degree of attitude and orbital coupling, such as low-thrust, fixed-thrust-direction spacecraft and very large spacecraft subject to higher-order gravitational torques and forces. A more theoretical direction for future research is the study of the relation of ERPs to the principal planes and angles of N-dimensional orientation and the interpretation of ERP singularities. Further research is also needed to investigate the in-between M situation, that is, $M \neq N(N-1)/2$. One straightforward approach is to pad the vector of extended Rodrigues parameters with additional fictitious coordinates until $M = N(N-1)/2$. The additional fictitious coordinates then represent constrained degrees of freedom.

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